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Design And Characterisation of Generalised Similarity: Notion of Oblique Symmetry

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Abstract

In the literature, few topics deal with the generality of geometric characterization of this application. Obviously, this is not a strange application. For a direct view, it is very easy to give a conjecture that it looks like the sum of a direct similarity and of an indirect similarity. However, by making some in-depth studies concerning the properties diffuse geometrics in this application this is not quite the case. Subsequently, this paper will explain some rich geometric properties provided by this application.

Keywords: *Generalized similarity, oblique and orthogonal symmetry, analytic and complex expressions, geometric characterization of a complex application.*

1. Introduction

Calculation is thus born with writing, or more exactly the first writing concerns the books of accounts, which give a "concrete" side. Geometry predates algebra by several thousand years (Houston (2009), ÉduSCOL (2009), E. Albert (1993), Maury (1994), Payot. (1913), POLYA. (1965), Richard (2005), Dumont (2004)). We agree to situate the transition between a geometry of pure form and a geometry subjected to the analysis of numbers between Egypt and Greece. Pythagoras, like the others, inherits the problems posed by geometers. Geometry is the first tangible form of abstract intelligence, and Egypt is its first high school (Shabert (1990), Malis et Vargas (2007), Semple et Kneebone (1952), Faugeras (1992)). She is the Science that studies built with sacred geometry. Categorically, the scientific method is always to find the determinisms acting on the creature, on the mathematical tools using the mathematical tools to simplify or generate our professional life. Subsequently, in the context of Euclidean geometry, several results are already discovered and already influencing the way designers design their activities (Shabert (1990), UNESCO (2011), Devlin (2012), Houston (2009)). Principles are added to clarify the numerous scientific results (Siegel (1989), Huybrechts (2005), Hartshorne (1977), Griffiths et Harris (1993), Fay (1993)). This is the reas on why in this paper, we will be interested in the generalized study of the characterization of the complex application f such as $f(z) = \mathcal{A}z + \mathcal{B}\bar{z} + \mathcal{C}$. Indeed, in the literature, thissubject remains still on the trail of investigative prospects. Presumably, his misleading expression gives us his as pect as the sum of direct and indirect similarity. However, after some extensive studies concerning its rich diffuse geometric properties in this one, several geometric and interesting results were revealed in this paper. These results were effectively revealed during a real culture by the manipulation of the compositions of some isometries and the conception of fixed point theorem (Jacquier (2012), Moakher (2002), Benhimane et Malis (2004), Malis et Chaumette (2002), Y. Fang et de Queiroz (2002)). The rest of this paper is organized as follows: the 2 section remender the notion of vectoriel space, the section 3 recalls some necessary tools in this theory; the section 4 concerns the notion of fixed point of the complex map *f*; the 5 section introduced the concept of oblique and orthogonal symmetry: complex and analytic expressions; the 6 section consists of talking about the inverse problem of the 5 section; the section 7 concerns the geometric properties of *f* apart from the proper oblique symmetry; the 8 section gives the applications and the 9 section presents a conclusion and perspectives.

2. Vector space

A vector space is a set whose elements, vectors, can be added and multiplied by scalars. Over a given field, vector spaces are classified by their dimension, by definition the cardinal of any basis. An affine space is informally a vector space for which the position of the null vector has been forgotten. This structure allows us to speak of linearity. More mathematically, we have:

Definition 1. *Let* $\mathbb{K}$ *be a field. We say* $(E,\oplus)$ *is a K-vector space if* $(E,\oplus)$*is an abelian group and if is an external law on E having* $\mathbb{K}$ *as operator domain, verifying the following four points :* $\forall(\lambda,\ \mu) \in \mathbb{K}^2, \forall(x,\ y) \in E^2$ *we have :*

i. $1_{\mathbb{K}}\otimes x = x$;

ii. $\lambda \otimes (x \oplus y) = (\lambda \otimes x) \oplus (\lambda \otimes y)$;

iii. $(\lambda \otimes \mu) \otimes x = \lambda \otimes (\mu \otimes x)$;

iv. $(\lambda \oplus \mu) \otimes x = (\lambda \otimes x) \oplus (\mu \otimes x)$.

Remark 1. *Note that* $M_{n,p}(\mathbb{K})$ *is the set of matrices n rows and p columns which constitute a vector space of dimension* np. *More specifically,* $M_n(\mathbb{K})$ *is the set of square matrices constituting* *a vector space of dimension* n^2*. Thus,* $M_2(\mathbb{K})$ *is the set of square matrices which constitute a vector space of dimension 4.*

Theorem 1. *Any free system consisting of n vectors in a vector space E of dimensions' n constitutes a basis of this space.*

3. Reminders on the complex presentation of plane transformations

Provide the plane P with an orthonormal reference have $(O,\ \vec{\imath},\ \vec{\jmath})$. Set $\mathbb{C}$ is the field of complex numbers. The following theorem constitutes the basic principle of this present theory (Dana-Picard (2003), Birkenhake et Lange (1992), Adler et Vanhaecke (2004)).

Theorem 2. *At any point M(a; b) of P, we can associate the complex number z = a+ib, where (a; b)*$\in \mathbb{R}^2$ *and reciprocally.*

This simply follows from the fact that the application:

$$\begin{aligned} g: \quad & P \to \mathbb{C} \\ & M(a,b) \mapsto z = a + ib \end{aligned}$$

(1)

is a bijection whose inverse bijection is:

$$\begin{aligned} g^{-1}: \quad & \mathbb{C} \to P \\ & z = a + ib \mapsto M(a,b) \end{aligned}$$

(2)

- **Vocabulary**
  - point M(*a*; *b*) is called the image of the complex number *z* = *a* + *ib*;
  - complex number *z* = *a* + *ib* is called the affix of the point M(*a*; *b*);
  - we often denote *z* = affixe(M) or *z*= aff(M);
  - to any complex number *z* = *a* + *bi* (with *a* and *b* reals), we can associate the vector $\vec{U}\begin{vmatrix} a \\ b \end{vmatrix}$.

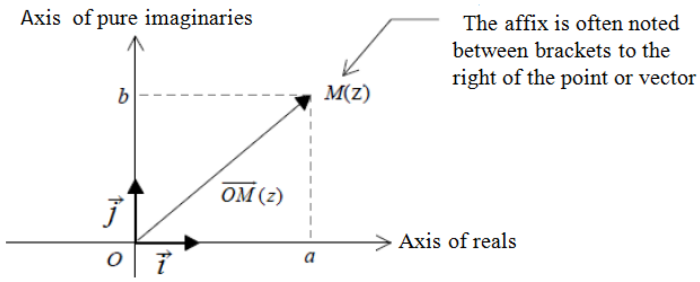

FIG. 1: Geometric representation of a complex number

Thus, we can then associate to any transformation h of the plan P a transformation $\check{h}$ of the field $\mathbb{C}$ on itself

according to the following commutative diagram:

$$\begin{array}{ccc} P & \overset{h}{\rightarrow} & P \\ g^{-1}\uparrow & = & \downarrow g \\ \square & \overset{h}{\rightarrow} & \square \end{array}$$

(3)

Diagram (3) confirms that $\check{h} = gohog^{-1}$ transforms the whole result of the algebraic calculation of $\mathbb{C}$ into P, and

$h = g^{-1}o\check{h}og$ all the geometric operations of the complex number plane (Saff et Snider (1993), Berger (1994)). Hence the following theorem.

Theorem 3. *With the notation of the commutative diagram (3), h is a one-to-one correspondence of the plane P on itself if, and only if,* $\check{h}$ *is a one-to-one correspondence from* $\mathbb{C}$ *to itself.*

Remark 2. 1. *Any bijective affine application h is associated with one and only one complex bijection* $\check{h}$.

2. The advantage of working in $\mathbb{C}$ *is the ability to make algebraic operations work defining the field* $\mathbb{C}$*, which is impossible to do in the plane P.*

3. Thus, one can transport a geometric problem of P in the field $\mathbb{C}$ *to constitute said geometry complex numbers.*

Hence the interest of the above theorem.

Taking into account the bijection between the applications g and g^{-1} and the commutative diagram (3), we will

see the analytical and complex expressions of the oblique symmetry which is diffuse in this complex application

f. In besides, we will relate the rich geometrical properties provided by *f*.

3.1. Application in mathematics

We consider here a normed vector space endowed with an orthonormal basis $(O,\ \vec{\imath},\ \vec{\jmath})$. In this space, an application is a relation between two sets of values for which each element of the first (called starting set or source) is linked to a single element of the second. The term competes with that of function, although this sometimes refers more specifically to applications whose purpose is a set of numbers and sometimes, on the contrary, encompasses more broadly the relations for which each element of the starting set is connected to at most one element of the target set. An application can have non-numeric values, like the one that associates each student in a class with their day of birth, or the application that each card of a deck of 32 cards associates its color. It can be injective or surjective depending on the uniqueness or the existence of an antecedent for each element of the target set. An application with these two properties is a bijection, which then admits a reciprocal application. Consequently, in an affine plane with an orthonormal frame $(O,\ \vec{\imath},\ \vec{\jmath})$ we define a complex application f by:

$$\begin{aligned} f:\ & \mathbb{C} \to \mathbb{C} \\ & z \mapsto f(z) = \mathcal{A}z + \mathcal{B}\bar{z} + C \end{aligned}$$

Notice that, in this paper, our work focuses on the geometric characterization of the application f as a result, we will immediately begin subsection (4) which will introduce the notion of its fixed point which is the axiomatic bas is of our study.

3.2. A few reminders about plane isometries

Definition 2. *In geometry, an isometry is a transformation that preserves lengths. She is then a case particular similarity. The term isometry is sometimes a bit vague. It can refer to two separate terms. An isometry may refer to:*

1. a vector isometry, it will then be more prudent to speak of unitary transformation or, if the space of departure and arrival are equal, of orthogonal automorphic;
2. an affine isometry, i.e. a one-to-one transformation of an affine Euclidean space into another which keeps distance. We generalize this notion to bijective transformations of a metric space in another that preserves distances.

Generally speaking, an application of the plane to itself is an isometry if it preserves distances. That is, if $M'N' = MN$ for all points M and N of the respective image plane M' and N'.

Theorem 4. *The identity of the plane, the translations, the orthogonal symmetries and the rotations are isometries. The images of two distinct points of the plane by an isometry are two distinct points.*

Theorem 5. *An application of the plane to itself is an isometry if and only if it preserves the product scalar. An application of the plane is said to be an isometry, if and only if* $\overrightarrow{AB}.\overrightarrow{AC} = \overrightarrow{A'B'}.\overrightarrow{A'C'}$ for all points A and C respective images A' and C'.

Theorem 6. *An isometry preserves the barycenter of two points. In particular, an isometry preserves the middle of a segment.*

Corollary 1.

a. Image of a line by an isometry is a line.

b. Image of a segment by an isometry is a segment which is isometric to it.

c. *Images of two parallel lines by an isometry are two parallel lines. We say that an isometry maintain parallelism.*

d. *Image of a parallelogram by an isometry is a parallelogram.*

e. *Images of two perpendicular lines by an isometry are two perpendicular lines (one says that an isometry preserves orthogonality).*

f. *Image of a circle by an isometry is a circle which is isometric to it.*

g. *Image by an isometry of the tangent at a point* M *to a circle is the tangent to the image circle, at point* M' *image of* M. *An isometry is said to preserve contact.*

3.3. Notations involved

In the following, next notations are put into play, and all the notations indexed by the letters *oq* could take on different values in each of the cases studied. In fact, it means « associated with symmetry any oblique ».

- A, *B*, C : the respective images of the $\mathcal{A}, \mathcal{B}, \mathcal{C}$ complex numbers such as A(a_1; b_1);*B*(a_2; b_2);C(x_0; y_0);
- the sum of the $\vec{U}_{A+B} = \overrightarrow{OA} + \overrightarrow{OB}$ vectors;
- $\mathcal{A}$: the complex number relating to the *f* application, such as $\mathcal{A} = a_1 + ib_1$;
- $\mathcal{A}_{ob}$: complex number relating to the f_{ob} application;
- $\mathcal{A}_{oq}$: complex number relating to the f_{oq} application, such as $\mathcal{A}_{oq} = 0 + ib_{q1}$: where $b_{q1} < 0$;
- $\mathcal{B}$: the complex number relating to the f application, such as $\mathcal{B} = a_2 + ib_2$;
- $\mathcal{B}_{ob}$: complex number relating to the f_{ob} application;
- $\mathcal{B}_{oq}$: complex number relating to the f_{oq} application, such as $\mathcal{B}_{oq} = a_{q2} + ib_{q2}$;
- $\mathcal{B}_{or}$: complex number relating to the f_{or} application;
- $\mathcal{C}$: complex number relating to the f application, such as C = $x_0 + iy_0$;
- $\mathcal{C}_{ob}$: complex number relating to the f_{ob} application;
- $\mathcal{C}_{oq}$: complex number relating to the f_{oq} application;
- $\mathcal{C}_{oq}$: complex number relating to the f_{or} application;
- Inv(*f*): set of invariant points of the *f* complex application;
- Inv(f_{ob}): set of invariant points of the f_{ob} complex application;
- Inv(f_{oq}): set of invariant points of the f_{oq} complex application;
- *HI*: identity of the complex application;
- f_p: application relative to positive isometry;

- f_n: application relative to negative isometry;
- f_{ob}: expression of the f application relative to the proper oblique symmetry;
- f_{oq}: expression of the f application relative to any fob oblique symmetry other than;
- f_{or} : application relating to orthogonal symmetry extracted from the fob application ;
- $\mathcal{H}_{(\omega_{oq},\rho_p)}$: homothety with ω_{oq} center and ρ_p ratio, with $\rho_p = \sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}$;
- $\mathcal{H}_{(\omega_{oq},\rho_n)}$: the homothety with ω_{oq} center and ρ_n ratio, with $\rho_n = \sqrt{|\mathcal{B}|^2 - |\mathcal{A}|^2}$;
- $K = M_{Sox}$: matrix relating to the orthogonal symmetry as for $x'Ox$;
- $\mathcal{L}(\mathbb{R}^2)$: set of lenear applications in $\mathbb{R}^2$;
- z: complex number, such as $z = x + iy$, with image's point $M(x, y)$;
- z': complex number, such as $z' = x' + iy'$, with image's point $M'(x', y')$;
- α and φ: two axiomatic angles associated with the f application;
- α_{ob} and φ_{ob} : two angles associated with the f_{ob} complex application;
- α_{or} and φ_{or} : two angles associated with the f_{or} application;
- α_{oq} and φ_{oq} : two angles associated with the f_{oq} application ;
- Δ_o : axis of an axiomatic of an oblique symmetry relative to the f complex application : $\Delta_o : ax + by + c = 0$;
- Δ_{ob}: axis of an oblique symmetry relative to the fob complex application;
- $\Delta_{\omega_{oq}//ox}$: axis of an orthogonal symmetry passing by ω_{oq} parallel to $x'Ox$;
- $\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)}$: axis of an oblique symmetry relative to the f_{oq} complex application passing by ω_{oq} making an angle $\frac{\theta}{2}$ as for the $x'Ox$;
- Δ_{oq}: axis of an oblique symmetry relative to the f_{oq} complex application forming an angle φ_{oq} as for the $x'Ox$ axis;
- Δ_{or}: axis of an orthogonal symmetry extracted from f_{ob};
- φ_{or}: angle associated with the f_{oq} complex application;
- ω: an invariant points of the f complex application;
- ω_{ob}: an invariant point of the fob complex application;
- ρ_n: ratio of $\mathcal{H}_{(\omega,\ \rho_n)}$ relative to $det M_{1+2} < 0$;
- ρ_p: ratio of $\mathcal{H}_{(\omega,\ \rho_p)}$ relative to $det M_{1+2} > 0$;
- ω_{oq}: an invariant point of the f_{oq} complex application.

- $S_{\left(\Delta_{(oq,\ \omega_{oq},//x'ox)}\right)}$: orthogonal symmetry as for $\Delta_{oq}$to the axis bay through ω_{oq} and parallel to the $x'Ox$ axis;
- $S_{\left(\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)}\right)}$: orthogonal symmetry as for Δ_{oq} axis passing by ω_{oq} and forming an angle $\frac{\theta_{oq}}{2}$ relative to the $x'Ox$ axis;
- *I, J, K, L*: matrix such as $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so that

TAB. 1: The two by two product to the left of these matrix

Starting from theorem 1, it is then clear that *I, J, K, L* constitute a basis of $M_2(\mathbb{R})$. Hence, any matrix $M \in M_2(\mathbb{R})$ can be written uniquely as a linear combination of *I, J, K, L*.

In effect,

- $\mathcal{R}_{(\theta_{oq},\omega_{oq})}$: rotation with θ_{oq} angle and ω_{oq}center. This matrix is $M_{\theta_{oq}} = \begin{pmatrix} sin\theta_{oq} & -cos\theta_{oq} \\ cos\theta_{oq} & sin\theta_{oq} \end{pmatrix}$ here matrix relating to this rotation so that $M_{\theta_{oq}} = Icos\theta_{oq} + Jsin\theta_{oq}$ with determinant 1;
- $M_{(1+2)}$: matrix relating to the linear part of the f application such as: $M_1 = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}$. In fact, $M_{(1+2)} = a_1I + b_1J + a_2K + b_2L$ of determinant : $detM_{(1+2)} = |\mathcal{A}|^2 - |\mathcal{B}|^2$.
- $M_{(q1+q2)}$: matrix relating to the oblique symmetry of expression fob such as: $M_{s1} = \begin{pmatrix} 0 & -b_{q1} \\ b_{q1} & 0 \end{pmatrix}$ and $M_{q2} = \begin{pmatrix} a_{q2} & -b_{q2} \\ b_{q2} & a_{q2} \end{pmatrix}$. Actually, $M_{(q1+q2)} = b_{q1}J + a_{q2}K + b_{q2}L$ of determinant : $detM_{(s1+s2)} = |\mathcal{A}|^2 - |\mathcal{B}|^2$; numerically is always equal -1;
- $M_{(s1+s2)}$: matrix relating to the oblique symmetry of expression fob such as: $M_{s1} = \begin{pmatrix} 0 & -b_{s1} \\ b_{s1} & 0 \end{pmatrix}$ and $M_{s2} = \begin{pmatrix} a_{s2} & -b_{s2} \\ b_{s2} & a_{s2} \end{pmatrix}$. Actually, $M_{(s1+s2)} = b_{s1}J + a_{s2}K + b_{s2}L$ of determinant : $detM_{(s1+s2)} = |\mathcal{A}|^2 - |\mathcal{B}|^2$, numerically is always equal -1;
- $M_{\alpha_{ob}\varphi_{ob}}$: matrix relating to the f_{ob} application;

Starting from the expression $M_{(1+2)} = a_1I + b_1J + a_2K + b_2L$, we see that the matrix $M_{(1+2)}$ (right part of this equality) represents a linear map and the quantity $a_1I + b_1J + a_2K + b_2L$ (left part) represents a generalized similarity. Hence the following theorem.

Theorem 7. Any mapping $f \in \mathcal{L}(\mathbb{R}^2)$ can be considered generalized similarity.

Indeed, let A, *B*, C be three points of *P* with respective affixes A = $a_1 + ib_1$, B = $a_2 + ib_2$, C = $x_0 + iy_0$, and $f(z) = z' = (a_1 + ib_1)z + (a_2 + ib_2)\bar{z} + x_0 + iy_0$ of analytic expression:

$$\begin{cases} x' = (a_1 + a_2)x + (-b_1 + b_2)y + x_0 = a_{11}x + a_{12}y + x_0 \\ y' = (b_1 + b_2)x + (a_1 - a_2)y + x_0 = a_{21}x + a_{22}y + y_0 \end{cases}$$

(4)

Moreover, if we put

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ b_2 & -a_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} a_{11} + a_{22} & a_{12} - a_{21} \\ -a_{12} + a_{21} & a_{11} + a_{22} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a_{11} - a_{22} & a_{12} + a_{21} \\ a_{12} + a_{21} & -a_{11} + a_{22} \end{pmatrix} \\ &= \frac{1}{2}(a_{11} + a_{22})I + \frac{1}{2}(-a_{12} + a_{21})J + \frac{1}{2}(a_{11} - a_{22})K + \frac{1}{2}(a_{12} + a_{21})L \end{aligned}$$

(5)

then,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mapsto \frac{1}{2}\left[(a_{11} + a_{22}) + i(-a_{12} + a_{21})\right]z + \frac{1}{2}\left[(a_{11} - a_{22}) + i(a_{12} + a_{21})\right]\bar{z} + x_0 + iy_0$$

4. Fixed point of f application

Let us denote by ω this point ; so generally speaking, its affix z_ω has the expression:

$$z_\omega = \frac{\mathcal{B}\bar{\mathcal{C}} + \mathcal{C}\left(1 - \bar{\mathcal{A}}\right)}{1 - 2\mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2}$$

(6)

The expression of the affix of this fixed point leads us to derive the following theorem (8).

Theorem 8. 1. *If* $1 - 2\mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 \neq 0$, *then this point is unique such as:* $z_\omega = \frac{\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}})}{1 - \mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2}$;

2. *if* $1 - 2\mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) = 0$ *then there is an infinity of fixed points ;*

3. *if* $1 - 2\mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) \neq 0$, *then this point does not exist.*

Corollary 2. *Any f complex application satisfying condition (2) admits the set of invariant points noted Inv such as:*

$$Inv(f): \begin{cases} \underbrace{x\left[1-\mathcal{Re}(\mathcal{A}+\mathcal{B})\right]+y\left[Im(\mathcal{A}-\mathcal{B})\right]=\mathcal{Re}(C)} \\ \Downarrow \\ \Uparrow \\ \overbrace{-x\left[Im(\mathcal{A}+\mathcal{B})\right]+y\left[1-\mathcal{Re}(\mathcal{A}-\mathcal{B})\right]=Im(C)} \end{cases}$$

These equations are homogeneous. One can then use one of them. However, we would again use the expression general rule if there is ambiguity on the choice of an invariant point.

$$Inv(f): x[1-\mathcal{Re}(\mathcal{A}+\mathcal{B})]+y[Im(\mathcal{A}-\mathcal{B})]-x[Im(\mathcal{A}+\mathcal{B})]+y[1-\mathcal{Re}(\mathcal{A}-\mathcal{B})]=\mathcal{Re}(\mathcal{C})+Im(\mathcal{C}).$$

Remark 3. *Notice that these three conditions constitute as axiomatic ideas of this present research.*

5. Oblique and orthogonal symmetry: complex and analytical expressions

In a plane *P*, we consider the line $(\Delta_0): ax+by+c=0$, $\vec{U}(-b,a)$, its direction vector so that $z_{\vec{U}}=\rho e^{i\varphi}$, where $sin\varphi=\frac{a}{\sqrt{a^2+b^2}}$, and $cos\varphi=\frac{-b}{\sqrt{a^2+b^2}}$, it is then obvious that, $(\Delta_0): xsin\varphi-ycos\varphi+\mu=0$, where $\mu=\frac{c}{\sqrt{a^2+b^2}}$. Line (Δ_0) makes an angle φ as for the $(O,\vec{\iota})$ axis, and the oblique symmetry is directed by vector $\vec{V}\big(cos(\varphi-\alpha), sin(\varphi-\alpha)\big)$ forming an angle φ as for the (Δ_0) axis (Cf. Figure 2).

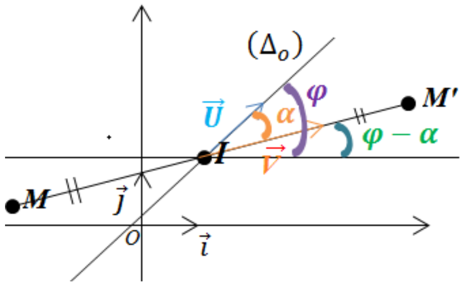

FIG. 2: Oblique symmetry directed by a vector as for to a line

As *I* is the middle of segment $[MM']$, it follows that $I\left(\frac{x'+x}{2}, \frac{y'+y}{2}\right)$, $I\in\Delta_0$.

Point $I\in\Delta_0$ is equivalent to:

$$(x' + x)sin\varphi - (y' + y)cos\varphi + 2\mu = 0$$
(7)

What's more $\overrightarrow{MM'} // \vec{V}$, then we have the equation:

$$(x' - x)sin(\varphi - \alpha) - (y' - y)cos(\varphi - \alpha) = 0$$
(8)

Two relations (7) and (8) leads us to have the expression (9) called the general analytical expression of the symmetry of the axis Δ_0 and the angle $\alpha = (\overrightarrow{MM}, \overrightarrow{U})$ defined by:

$$\begin{cases} x' = -\frac{1}{\sin\alpha}\left[\sin(2\varphi-\alpha)x - (\cos\alpha + \sin(2\varphi-\alpha))y + 2\mu\cos(\varphi-\alpha)\right] \\ y' = -\frac{1}{\sin\alpha}\left[(\cos\alpha - \sin(2\varphi-\alpha))x - \sin(2\varphi-\alpha)y + 2\mu\sin(\varphi-\alpha)\right] \end{cases}$$
(9)

Or, in matrix form, we have:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = -\frac{1}{\sin\alpha}\begin{pmatrix} \sin(2\varphi-\alpha) & -(\cos\alpha + \sin(2\varphi-\alpha)) \\ (\cos\alpha - \sin(2\varphi-\alpha)) & -\sin(2\varphi-\alpha) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{\sin\alpha}\begin{pmatrix} 2\mu\cos(\varphi-\alpha) \\ 2\mu\sin(\varphi-\alpha) \end{pmatrix}$$
(10)

From the expression (9), we can get the complex expression (11) defined by:

$$\begin{aligned} z' &= -i(\cot\alpha)z - \frac{1}{\sin\alpha}\left[\sin(2\varphi-\alpha) - i\cos(2\varphi-\alpha)\right]\bar{z} - \frac{2\mu}{\sin\alpha}\left[\cos(\varphi-\alpha) + i\sin(\varphi-\alpha)\right] \\ &= -i(\cot\alpha)z + \frac{1}{\sin\alpha}e^{i\left(\frac{\pi}{2}+2\varphi-\alpha\right)} - \frac{2\mu}{\sin\alpha}e^{i(\varphi-\alpha)} \end{aligned}$$
(11)

Either still form as written on the title:

$$f_{ob}(z) = \mathcal{A}_{ob}z + \mathcal{B}_{ob}\bar{z} + \mathcal{C}_{ob}$$
(12)

Theorem 9. *For any f application, if A is pure negative imaginary and* $1 + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) = 0$, *then f becom* f_{ob} *which represents an oblique symmetry generated by the line* $(\Delta_0) : ax + by + c = 0$ *which is equivalent to* $(\Delta_{ob}) : \left(x - x_{z_{\omega_{ob}}}\right)sin\varphi_{ob} - \left(y - y_{z_{\omega_{ob}}}\right)cos\varphi_{ob} = 0$ *making an angle* φ_{ob} *by relative to the* $(O, \vec{\imath})$ *axis, directed by a vector* $\overrightarrow{V_{ob}}(cos(\varphi_{ob} - \alpha_{ob}),\ sin(\varphi_{ob} - \alpha_{ob}))$ *forming an angle* α_{ob} *as for to an axis* $(\Delta_{ob}) : \left(x - x_{z_{\omega_{ob}}}\right)sin\varphi_{ob} - \left(y - y_{z_{\omega_{ob}}}\right)cos\varphi_{ob} = 0$, *with analytical expression (13) and with complex expression (15).*

$$\begin{cases} x' = -\frac{1}{\sin\alpha}\left[\sin(2\varphi-\alpha)x - (\cos\alpha + \sin(2\varphi-\alpha))y + 2\mu\cos(\varphi-\alpha)\right] \\ y' = -\frac{1}{\sin\alpha}\left[(\cos\alpha - \sin(2\varphi-\alpha))x - \sin(2\varphi-\alpha)y + 2\mu\sin(\varphi-\alpha)\right] \end{cases}$$

(13)

Let be:

$$M_{\alpha_{ob}\varphi_{ob}} = -\frac{1}{\sin\alpha}\begin{pmatrix} \sin(2\varphi_{ob}-\alpha_{ob}) & -(\cos\alpha_{ob} + \sin(2\varphi_{ob}-\alpha_{ob})) \\ (\cos\alpha_{ob} - \sin(2\varphi_{ob}-\alpha_{ob})) & -\sin(2\varphi_{ob}-\alpha_{ob}) \end{pmatrix}$$

(14)

$$z' = -i(\cot\alpha_{ob})z - \frac{1}{\sin\alpha_{ob}}\left[\sin(2\varphi_{ob}-\alpha_{ob}) - i\cos(2\varphi_{ob}-\alpha_{ob})\right]\bar{z} - \frac{2\mu}{\sin\alpha_{ob}}\left[\cos(\varphi_{ob}-\alpha_{ob}) + i\sin(\varphi_{ob}-\alpha_{ob})\right]$$
$$= -i(\cot\alpha_{ob})z + \frac{1}{\sin\alpha_{ob}}e^{i\left(\frac{\pi}{2}+2\varphi_{ob}-\alpha_{ob}\right)} - \frac{2\mu}{\sin\alpha_{ob}}e^{i(\varphi_{ob}-\alpha_{ob})}$$

(15)

Theorem 10. Any f_{ob} application admits a set of invariant points with equation $Inv(f_{ob})$ such as:

$$Inv(f_{ob}):\ x[1 - \mathcal{R}e(\mathcal{B}_{ob})] + y[Im(\mathcal{A}_{ob} - \mathcal{B}_{ob})] = \mathcal{R}e(\mathcal{C}_{ob});$$

$$Inv(f_{ob}):\ -x[Im(\mathcal{A}_{ob} + \mathcal{B}_{ob})] + y[1 - \mathcal{R}e(\mathcal{B}_{ob})] = Im(\mathcal{C}_{ob});$$

$$Inv(f_{ob}): x[1 - \mathcal{R}e(\mathcal{B}_{ob}) + Im(\mathcal{A}_{ob} + \mathcal{B}_{ob})] + y[1 - \mathcal{R}e(\mathcal{B}_{ob}) - Im(\mathcal{A}_{ob} - \mathcal{B}_{ob})] = \mathcal{R}e(\mathcal{C}_{ob}) + Im(\mathcal{C}_{ob}).$$

Proposition 1. *For any invariant point* $Inv(f_{ob})$, *we can choose an invariant point* ω_{ob} *extracted from this line as a center put into play during all the operations carried out.*

Remark 4. *Notice that it is very easy to verify that the last expression (11) indeed satisfies the three relations if* $1 - 2\mathcal{R}e(\mathcal{A}_{ob}) + |\mathcal{A}_{ob}|^2 - |\mathcal{B}_{ob}|^2 = 0$ *and* $\mathcal{B}_{ob}\overline{\mathcal{C}_{ob}} - \mathcal{C}_{ob}(1 - \overline{\mathcal{A}_{ob}}) = 0$ *and* $\mathcal{R}e(\mathcal{A}_{ob}) = 0$, *those which constitute as necessary conditions f to be* f_{ob}.

Theorem 11. *Any oblique symmetry is always characterized by its direction vector* $\overrightarrow{V_{ob}}$, *its axis of symmetry* (Δ_{ob}) : $(x - x_{\omega_{ob}})sin\varphi_{ob} - (y - y_{\omega_{ob}})cos\varphi_{ob} = 0$ *and its angle* α_{ob} : *such as* $\alpha_{ob}(\overrightarrow{V_{ob}}, \overrightarrow{U_{ob}})$ *where* $\overrightarrow{U_{ob}}$ $(cos\alpha_{ob}, sin\alpha_{ob})$ *is the director vector of* (Δ_{ob}) *(Cf. figure 10).*

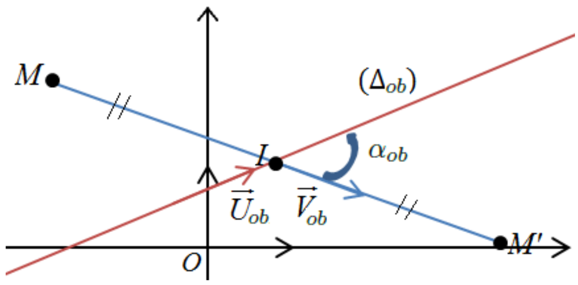

FIG. 3: Oblique symmetry

Remark 5. *These three cases lead us to have a method allowing to find the coefficients a, b, c generating the*

expression of $(\Delta_0) : ax + by + c = 0$ *if the* f_{ob} *application is given beforehand.*

6. Inverse problem (of section 5)

We already think of answering the question: given the complex expression of a relative symmetry to the f_{ob} application of $(\Delta_0) : ax + by + c = 0$ axis, then « how can we calculate the coefficients *a*, *b*, *c* allowing to give the expression of $(\Delta_0) : ax + by + c = 0$? ». In general, it is a bit cumbersome to work with the direct search for these coefficients. Notice that the three conditions in **Remark 4** confirm that fob admits an infinity of fixed points. This means that it admits a set denoted $Inv(f_{ob})$ of invariant points such as $Inv(f_{ob}): x[1 - \mathcal{R}e(\mathcal{B}_{ob}) + Im(\mathcal{A}_{ob} + \mathcal{B}_{ob})] + y[1 - \mathcal{R}e(\mathcal{B}_{ob}) - Im(\mathcal{A}_{ob} - \mathcal{B}_{ob})] = \mathcal{R}e(\mathcal{C}_{ob}) + Im(\mathcal{C}_{ob})$. From this equation, we can easily choose ω_{ob} belongs to line. Assuming that the line (Δ_0) passes through ω_{ob} and parallel to the vector $\overrightarrow{U_{ob}}\,(cos\alpha_{ob}, sin\alpha_{ob})$, it follows that this line is equivalent to $(\Delta_{ob}) : (x - x_{\omega_{ob}})sin\varphi_{ob} - (y - y_{\omega_{ob}})cos\varphi_{ob} = 0$. With the expression for f_{ob}, we can easily determine the angle α_{ob} and φ_{ob} to structure the equation of (Δ_{ob}) according to the two following relations:

$$\begin{cases} |\mathcal{B}_{ob}| = \dfrac{1}{\sin \alpha_{ob}} \\ \arg \mathcal{B}_{ob} = \dfrac{\pi}{2} + 2\varphi_{ob} - \alpha_{ob} \end{cases}$$

(16)

We now have a method to find the equation of the line (Δ_{ob}) if the complex expression of the oblique symmetry is already given previously in the form (15).

In fact, these three relations lead us to conclude that:

$$\begin{cases} \sin \alpha_{ob} = \dfrac{1}{|\mathcal{B}_{ob}|} \\ \varphi_{ob} = \dfrac{1}{2}\left(\arg \mathcal{B}_{ob} - \dfrac{\pi}{2} + \alpha_{ob} \right) \end{cases}$$

(17)

Corollary 3. *If* $\alpha_{ob} = k\pi$, *where* $k \in \mathbb{Z}$, *then the oblique symmetry does not exist. However, if* $\alpha_{ob} = \frac{\pi}{2}$, *then we obtain orthogonal symmetry.*

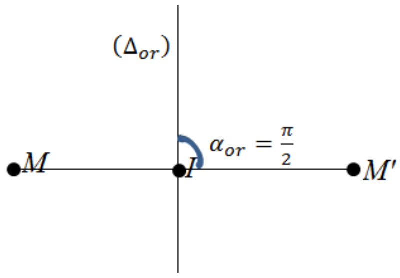

FIG. 4: Orthogonal axis symmetry (Δ_{ob})

On this subject, compared to the results above, more particularly **Theorem 9**, one can immediately draw the subsequent inductions.

Expression (9) becomes (18) defined by:

$$\begin{cases} x' = x\cos 2\varphi_{or} + y\sin 2\varphi_{or} - 2\mu \sin \varphi_{or} \\ y' = x\sin 2\varphi_{or} - y\cos 2\varphi_{or} + 2\mu \cos \varphi_{or} \end{cases}$$

(18)

Or, in matrix form, we have:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 2\varphi_{or} & \sin 2\varphi_{or} \\ \sin 2\varphi_{or} & -\cos 2\varphi_{or} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2\mu \begin{pmatrix} -\sin \varphi_{or} \\ \cos \varphi_{or} \end{pmatrix}$$

(19)

On this subject, we have the analytic expression of axis orthogonal symmetry Δ defined on (18) above. Through elsewhere, the expression (18) leads us to have the expression (20) called the complex expression of the said symmetry defined by:

$$\begin{aligned} z' &= \left(\cos 2\varphi_{or} + i\sin 2\varphi_{or}\right)\bar{z} + 2i\mu\left(\cos \varphi_{or} + i\sin \varphi_{or}\right) \\ &= e^{i2\varphi_{or}}\bar{z} + 2\mu e^{i\varphi_{or}} \end{aligned}$$

(20)

From where,

$$f_{or}(z) = e^{i2\varphi_{or}}\bar{z} + 2\mu e^{i\varphi_{or}}$$

Finally, the expression (17) can be written in forms (21) defined by:

$$\begin{cases} \alpha_{or} = \dfrac{\pi}{2} \\ \varphi_{or} = \dfrac{1}{2}\arg B_{or} \end{cases}$$

(21)

Theorem 12. Any orthogonal symmetry generated by the line $(\Delta_0) : ax + by + c = 0$, has for expressions analytical and complex in forms (23) or (23) and (24) or (25) defined by:

$$\begin{cases} x' = \dfrac{1}{a^2+b^2}\left[\left(a^2-b^2\right)x + 2aby + 2ac\right] \\ y' = \dfrac{1}{a^2+b^2}\left[2abx - \left(a^2-b^2\right)y + 2bc\right] \end{cases}$$

(22)

i.e.

$$\begin{aligned} z' &= \frac{1}{a^2+b^2}\left[\left(a^2-b^2+2iab\right)\bar{z} + 2ac + 2ibc\right] \\ &= \mathcal{B}_{or}\bar{z} + C_{or} \end{aligned}$$

(23)

Or

$$\begin{cases} x' = x\cos 2\varphi_{or} + y\sin 2\varphi_{or} - 2\mu\sin\varphi_{or} \\ y' = x\sin 2\varphi_{or} - y\cos 2\varphi_{or} + 2\mu\cos\varphi_{or} \end{cases}$$

(24)

Or

$$\begin{aligned} z' &= \left(\cos 2\varphi_{or} + i\sin 2\varphi_{or}\right)\bar{z} + 2i\mu\left(\cos\varphi_{or} + \sin\varphi_{or}\right) \\ &= e^{i2\varphi_{or}}\bar{z} + 2\mu^{\left(\varphi_{or}+\frac{\pi}{2}\right)} \end{aligned}$$

(25)

Remark 6. *Using the expression (24), we can easily find the coefficients (a; b; c) using the following relations:*

$$\begin{cases} \mathcal{Re}(\mathcal{B}_{or}) = -\dfrac{a^2-b^2}{a^2+b^2} \\ Im(\mathcal{B}_{or}) = -\dfrac{2ab}{a^2+b^2} \\ \mathcal{Re}(C_{or}) = -\dfrac{2ac}{a^2+b^2} \\ Im(C_{or}) = -\dfrac{2bc}{a^2+b^2} \end{cases}$$

(26)

Remark 7. *Notice that we can also display the expression of the line* (Δ_{or}) *generating the expression (20) which is equivalent to* $(\Delta_0) : ax + by + c = 0$ *if the expression (20) is available using the relations* $\frac{|\mathbb{C}|}{2}$ *and* $\varphi_{or} = \frac{1}{2} arg\, \mathcal{B}_{or}$ *so that* (Δ_{or}): $xsin\varphi_{or} + yxcos\varphi_{or} = 0$.

Example 1. It is very easy to check that the analytic and complex expressions of the orthogonal symmetry generated

by the line (Δ_0): $y = 2x + 3$ are such as:

$$\begin{cases} x' = -\frac{1}{5}[3x - 4y + 12] \\ y' = -\frac{1}{5}[-4x - 3y - 6] \end{cases}$$

(27)

Or

$$z' = -\frac{1}{5}\left[(3-4i)\bar{z} + 12 - 6i\right]$$

(28)

7. Geometric properties of *f* apart from proper oblique symmetry

In this section, we are interested in geometrically characterizing the f application, at the margin of f_{ob}, according to the possible cases concerning the situations of determinants of $M_{(1+2)}$ and fixed point of *f*.

7.1. Trivial case $\mathcal{A} = 0$ or $\mathcal{B} = 0$

Theorem 13. 1. *If* $\mathcal{B} = 0$ *and* $\mathcal{A} \neq 0$, *then this application represents a direct similarity.*

3. If $\mathcal{A} = 0$ *and* $\mathcal{B} \neq 0$, *then this application represents an indirect similarity.*

In fact, the proof is obvious by already knowing the rich geometric properties regarding these two transformations namely, if $\mathcal{A} = 0$, then $z_\omega = \frac{\mathcal{B}\,\bar{\mathcal{C}}+\mathcal{C}}{1-\mathcal{B}\,\bar{\mathcal{B}}}$, and if $\mathcal{B} = 0$, then $z_\omega = \frac{\mathcal{C}}{1-\mathcal{A}}$. The two expressions represent respectively the center of indirect similarity and that of the direct similarity. In what follows, our study consists in examining the cases where $\mathcal{A} \neq 0$ and $\mathcal{B} \neq 0$.

7.2. Case where $\boldsymbol{detM_{(1+2)}} < 0$

This case confirms that $M_{(1+2)}$ is decomposing into a homothetic with _n positive ratio and two isometries one of which is positive and the other is negative. That is to say *f* can be written in the form : $f = f_p o f_n$, where $f_p = \mathcal{H}_{(\omega_{oq},\rho_p)} o \mathcal{R}_{(\theta_{oq},\omega_{oq})}$ and $f_n = f_{oq}$. Using the expression of $M_{(1+2)}$ and $M_{(q1+q2)}$, this composition is equivalent to $M_{(1+2)} = \rho_n \times M_{\theta_{oq}} \times M_{(q1+q2)}$ where:

$$
\begin{aligned}
a_1 I + b_1 J + a_2 K + b_2 L &= \rho_n \left(I\cos\theta_{oq} + J\sin\theta_{oq} \right)\left(Jb_{q1} + Ka_{q2} + Lb_{q2} \right) \\
&= \rho_n [-Ib_{q1}\sin\theta_{oq} + Jb_{q1}\cos\theta_{oq} + K\left(a_{q2}\cos\theta_{oq} - b_{q2}\sin\theta_{oq} \right) \\
&\quad + L\left(b_{q2}\cos\theta_{oq} + a_{q2}\sin\theta_{oq} \right)]
\end{aligned}
$$

(29)

Indeed, this equation is of unknowns $\rho_n, b_{q1}, a_{q2}, b_{q2}, \theta_{oq}$. So that, by identification, we have:

$$
\begin{cases}
b_{q1}\sin\theta_{oq} = -\dfrac{a_1}{\rho_n} \\
b_{q1}\cos\theta_{oq} = \dfrac{b_1}{\rho_n} \\
a_{q2}\cos\theta_{oq} - b_{q2}\sin\theta_{oq} = \dfrac{a_2}{\rho_n} \\
a_{q2}\sin\theta_{oq} + b_{q2}\cos\theta_{oq} = \dfrac{a_1}{\rho_n}
\end{cases}
$$

(30)

The relation (30) leads us to have the relation (31) defined by:

$$
\begin{cases}
\rho_n = \sqrt{|\mathcal{B}|^2 - |\mathcal{A}|^2} \\
b_{q1} = -\dfrac{|\mathcal{A}|}{\sqrt{|\mathcal{B}|^2 - |\mathcal{A}|^2}} \\
\sin\theta_{oq} = \dfrac{\mathcal{Re}(\mathcal{A})}{|\mathcal{A}|} \\
\cos\theta_{oq} = -\dfrac{Im(\mathcal{A})}{|\mathcal{A}|} \\
a_{q2} = -\dfrac{\mathcal{Re}(\mathcal{B})\, Im(\mathcal{A}) - Im(\mathcal{B})\,\mathrm{Re}(\mathcal{A})}{\rho_n |\mathcal{A}|} \\
a_{q2} = -\dfrac{\mathcal{Re}(\mathcal{A})\,\mathcal{Re}(\mathcal{B}) + Im(\mathcal{A})\, Im(\mathcal{B})}{\rho_n |\mathcal{A}|}
\end{cases}
$$

(31)

What's more,

$$
\begin{aligned}
\mathcal{A}\overline{\mathcal{B}} &= \left[\mathcal{Re}(\mathcal{A}) + iIm(\mathcal{A}) \right]\left[\mathcal{Re}(\mathcal{B}) - iIm(\mathcal{B}) \right] \\
&= \mathcal{Re}(\mathcal{A})\mathcal{Re}(\mathcal{B}) + Im(\mathcal{A})Im(\mathcal{B}) + i\left[\mathcal{Re}(\mathcal{B})Im(\mathcal{A}) - \mathcal{Re}(\mathcal{A})Im(\mathcal{B}) \right]
\end{aligned}
$$

(32)

On the other hand,

$$\cos\theta_{oq} + i\sin\theta_{oq} = -\frac{Im(\mathcal{A})}{|\mathcal{A}|} + i\frac{\mathcal{R}e(\mathcal{A})}{|\mathcal{A}|} = i\left(\frac{\mathcal{R}e(\mathcal{A})}{|\mathcal{A}|} + i\frac{Im(\mathcal{A})}{|\mathcal{A}|}\right)$$

(34)

The three relations (31), (32) and (33) lead us to have the relations (35) defined by:

$$\begin{cases} \rho_n = \sqrt{|\mathcal{B}|^2 - |\mathcal{A}|^2} \\ \sin\theta_{oq} = \frac{\mathcal{R}e(\mathcal{A})}{|\mathcal{A}|} \\ \cos\theta_{oq} = -\frac{Im(\mathcal{A})}{|\mathcal{A}|} \\ \theta_{oq} = \arg(\mathcal{A}) + \frac{\pi}{2} \\ a_{q2} = -\frac{Im(\mathcal{A}\overline{\mathcal{B}})}{|\mathcal{A}|\sqrt{|\mathcal{B}|^2 - |\mathcal{A}|^2}} \\ b_{q2} = -\frac{\mathcal{R}e(\mathcal{A}\overline{\mathcal{B}})}{|\mathcal{A}|\sqrt{|\mathcal{B}|^2 - |\mathcal{A}|^2}} \end{cases}$$

(35)

A- Case where *f* admits a fixed point

In the case where *f* admits a fixed point z_ω $(1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 \neq 0)$, and given the decomposition of $M_{(1+2)} = \rho_n \times M_{\theta_{oq}} \times M_{(q1+q2)}$ resulting in the result of expression (36), then we have:

$$z' - z_\omega = \mathcal{A}(z - z_\omega) + \mathcal{B}\overline{(z - z_\omega)}$$

(36)

Let us recall in passing that

$$\begin{cases} \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = \rho_n\left(z - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ \mathcal{R}_{(\theta_{oq},\omega_{oq})}(z) = e^{i\theta_{oq}}\left(z - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ f_{oq}(z) = ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}} \end{cases}$$

(37)

Then,

$$\begin{cases} \mathcal{R}_{(\theta_{oq},\omega_{oq})} o f_{oq}(z) = e^{i\theta_{oq}}\left(f_{oq}(z) - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ \mathcal{H}_{(\omega_{oq},\rho_n)} o \mathcal{R}_{(\theta_{oq},\omega_{oq})} o f_{oq}(z) = \rho_n\left(e^{i\theta_{oq}}\left(f_{oq}(z) - z_{\omega_{oq}}\right) + z_{\omega_{oq}} - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \end{cases}$$

(38)

Given this relationship, for $\omega_{oq} = \omega$ we have:

$$\rho_n \left[\underbrace{e^{i\theta_{oq}} \left[\underbrace{ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}}}_{f_{oq}} - z_{\omega_{oq}} \right] + z_{\omega_{oq}} - z_{\omega_{oq}}}_{\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})(z)} \right] + z_{\omega_{oq}}$$

$$\underbrace{\qquad\qquad\qquad}_{\mathcal{H}_{(\omega_{oq},\rho_n)}\left[\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})\right](z)}$$

(39)

From where:

$$f(z) = \mathcal{H}_{(\omega_{oq},\rho_n)} o \mathcal{R}_{(\theta_{oq},\omega_{oq})} o f_{oq}(z)$$

Theorem 14. *If* $detM_{(1+2)} < 0$, *then for any given f complex application, the two angles* α_{oq} *and* φ_{oq} *associated with the* f_{oq} *complex application which is extracted from f are such as* $\alpha_{oq} = \frac{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}}{|\mathcal{A}|}$ and $\varphi_{oq} = \frac{\alpha}{2} + \frac{1}{2} arg(\bar{\mathcal{A}}\mathcal{B}) - \frac{\pi}{2}$.

Theorem 15. *If* $detM_{(1+2)} < 0$ 0 *and* z_ω $(1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 \neq 0)$ *exists and is unique, then the f complex application is the composite of* f_{oq} *oblique symmetry directed by the vector* $\overrightarrow{V_{oq}}\left(cos(\varphi_{oq} - \alpha_{oq}),\ sin(\varphi_{oq} - \alpha_{oq})\right)$ *which makes the angle* α_{oq} *such as sin* $\alpha_{oq} = \frac{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}}{|\mathcal{A}|}$ *as for to an axis* Δ_{oq} *making the angle* φ_{oq} *as for to an axis* $(O,\ \vec{\imath})$ *by the rotation of the center* ω_{oq}, *the angle* θ_{oq} *such as* $\theta_{oq} = arg(\mathcal{A}) + \frac{\pi}{2}$ *and by homothetic of the center* ω_{oq} *and the ratio* ρ_n.

Example 2. *Let f be the complex application defined by :* $f(z) = (1 + i\sqrt{3})z + [-\sqrt{3} - 1 + i(\sqrt{3} - 1)]\bar{z} + 2 - \sqrt{3} - i\sqrt{3}$. *It is very easy to verify graphically that f is the composite of the oblique symmetry* f_{oq} *defined by:* $f_{oq}(z) = -iz + (1 - i)\bar{z}$ *directed by the vector* $\vec{V}(0,\ 1)$ *which makes the angle* $\alpha_{oq} = \frac{\pi}{4}$ *relative to* $\Delta_{oq} : y = x$ *axis, making the angle* $\varphi_{oq} = \frac{\pi}{4}$ *relative to* $(O,\ \vec{\imath})$ *axis, by the rotation of center* $\omega_{oq}(1, 1)$ *and of angle* $\theta_{oq} = \frac{\pi}{6}$ *and the homothetic with ratio* $\rho_n = 2$ *and with center* ω_{oq}.

It is very easy to verify directly that the image of point $M_1(-1,\ 1)$ is point $M'_1(1 + 2\sqrt{3},\ 3) = f(M_1)$. This result is well proven in Figure 5. It can also be verified using expressions (39). This result is well proven in Figure 5. It can also be verified using expressions (40).

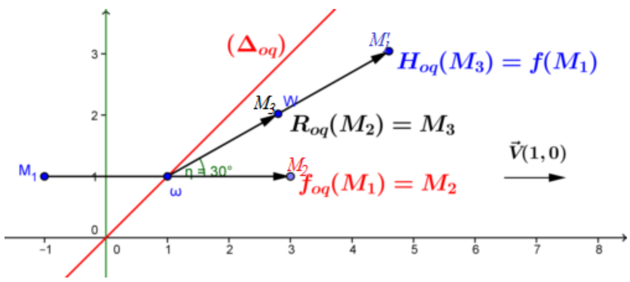

FIG. 5: Figure corresponding to Example 2.

$$\begin{cases} \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = 2z - 1 - i \\ \mathcal{R}_{(\omega_{oq},\rho_n)}(z) = \frac{1}{2}\left[\left(\sqrt{3}+i\right)z + 3 - \sqrt{3} + i\left(1-\sqrt{3}\right)\right] \\ f_{oq}(z) = -iz + (-1+i)\bar{z} \quad \text{of axis } y = x \text{ ; } \vec{V}(0,1) \end{cases}$$

(40)

From where, $f(z) = \mathcal{H}_{(\omega_{oq},\ \rho_n)}\ o\ \mathcal{R}_{(\omega_{oq},\theta_{oq})} o f_{oq}(z)$.

B- Case where $\mathbf{1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0}$ *and* $\mathbf{\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) = 0}$

As we have already announced in paragraph 4, these two cases tell us that the f complex application admits an infinity of fixed points. It is then clear from **Corollary 2** that f has the set of points invariants of which $Inv(f_{oq})$ defined by $nv(f): x[1 - \mathcal{R}e(\mathcal{A} + \mathcal{B})] + y[Im(\mathcal{A} - \mathcal{B})] - x[Im(\mathcal{A} + \mathcal{B})] + y[1 - \mathcal{R}e(\mathcal{A} - \mathcal{B})] = \mathcal{R}e(\mathcal{C}) + Im(\mathcal{C})$. We can then choose one of this bridge whose ω_{oq}. Which means that:

$$z' - z_{\omega_{oq}} = \mathcal{A}\left(z - z_{\omega_{oq}}\right) + \mathcal{B}\overline{\left(z - z_{\omega_{oq}}\right)}$$

(41)

$$\underbrace{\rho_n\left[\underbrace{e^{i\theta_{oq}}\left[\underbrace{ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}}}_{f_{oq}} - z_{\omega_{oq}}\right] + z_{\omega_{oq}}}_{\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})(z)} - z_{\omega_{oq}}\right] + z_{\omega_{oq}}}_{\mathcal{H}_{(\omega_{oq},\rho_n)}\left[\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})\right](z)}$$

(42)

From where:

$$f(z) = \mathcal{H}_{(\omega_{oq},\rho_n)} o \mathcal{R}_{(\theta_{oq},\omega_{oq})} o f_{oq}(z)$$

Theorem 15. *If* $detM_{(1+2)} < 0$ *and* z_ω *and* $1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) = 0$, *then the f complex application is the composite of* f_{oq} *oblique symmetry directed by the vector* $\overrightarrow{V_{oq}}\left(cos(\varphi_{oq}\alpha_{oq}),\ sin(\varphi_{oq} - \alpha_{oq})\right)$ *which makes the angle* α_{oq} *such as*

$\sin \alpha_{oq} = \frac{\sqrt{|\mathcal{A}|^2-|\mathcal{B}|^2}}{|\mathcal{A}|}$ as for to an axis Δ_{oq} making the angle φ_{oq} as for to an axis $(O,\ \vec{\imath})$ by the rotation of the center ω_{oq}, the angle θ_{oq} such as $\theta_{oq} = arg(\mathcal{A}) + \frac{\pi}{2}$ and by homothetic of the center ω_{oq} and the ratio ρ_n.

Example 3. *Let f be the complex application defined by : $f(z) = iz + (1+i)\bar{z} + 2i$. It is very easy to verify graphically that f is the composite of the oblique symmetry f_{oq} defined by: $f_{oq}(z) = -iz + (-1-i)\bar{z} - 4 - 4i$ directed by the vector $\vec{V}(-1,-1)$ which makes the angle $\alpha_{oq} = \frac{\pi}{4}$ relative to $\Delta_{oq} : x + 2 = 0$ axis, making the angle $\varphi_{oq} = \frac{\pi}{4}$ relative to $(O,\ \vec{\imath})$ axis, by the rotation of center $\omega_{oq}(1,1)$ and of angle $\theta_{oq} = \pi$ and the homothetic with ratio $\rho_n = 1$ and with center ω_{oq}.*

It is very easy to verify directly that the image of point $M_0(0,\ 0)$ is point $M'_0(0,\ 2) = f(M_0)$. This result is well proven in Figure 6. It can also be verified using expressions (43).

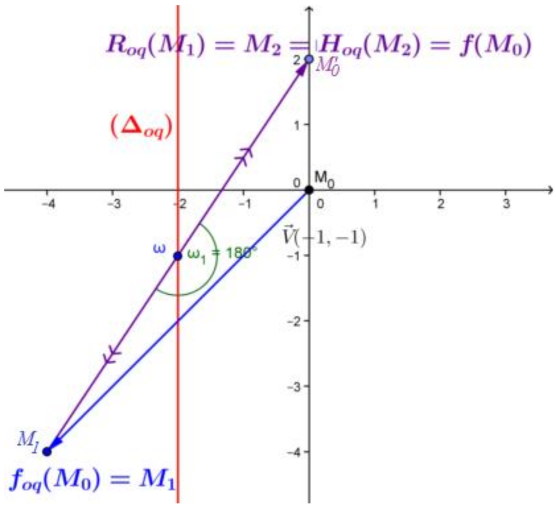

FIG. 6: Figure corresponding to Example 3.

$$\begin{cases} \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = z-1-i \\ \mathcal{R}_{(\omega_{oq},\rho_n)}(z) = -z-4-2i \\ f_{oq}(z) = -iz+(-1-i)\bar{z}-4-4i \quad \text{of axis } x+2=0\,;\vec{V}(-1,-1) \end{cases}$$
(43)

From where, $f(z) = \mathcal{H}_{(\omega_{oq},\ \rho_n)}\ o\ \mathcal{R}_{(\omega_{oq},\theta_{oq})} o f_{oq}(z)$.

C- Case where $\mathbf{1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0}$ ***and*** $\mathbf{\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1-\bar{\mathcal{A}}) \neq 0}$

Paragraph 4 has already confirmed that this case corresponds to the non-existence of a fixed point of the corresponding f complex application. In this case, we are going to construct a complex constant $\mathcal{C}_o$ so that $\mathcal{B}\bar{\mathcal{C}_o} - \mathcal{C}_o(1-\bar{\mathcal{A}}) = 0$. It is then clear from **Corollary 2** that f has the set of invariant points whose $nv(f)$ defined by $nv(f): x[1-\mathcal{R}e(\mathcal{A}+\mathcal{B})] + y[Im(\mathcal{A}-\mathcal{B})] - x[Im(\mathcal{A}+\mathcal{B})] + y[1-\mathcal{R}e(\mathcal{A}-\mathcal{B})] = \mathcal{R}e(\mathcal{C}_o) + Im(\mathcal{C}_o)$. We can then choose one of this bridge whose ω_{oq}. Hence the theorem following.

Theorem 17. *For any f complex application, if* $1-2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1-\bar{\mathcal{A}}) \neq 0$, *then there exists* $\mathcal{C}_o = ik(1+\mathcal{B}-\mathcal{A})$ *where* $k \in \mathbb{R}^*$ *verifying* $\mathcal{B}\bar{\mathcal{C}_o} - \mathcal{C}_o(1-\bar{\mathcal{A}}) = 0$ *such as* $f(z) = \mathcal{A}z + \mathcal{B}\bar{z} + \mathcal{C}_o + \mathcal{C} - \mathcal{C}_o$.

Given the expression of $f(z) = \mathcal{A}z + \mathcal{B}\bar{z} + \mathcal{C}_o + \mathcal{C} - \mathcal{C}_o$ we have:

$$z' - z_{\omega_{oq}} = \mathcal{A}\left(z - z_{\omega_{oq}}\right) + \mathcal{B}\overline{\left(z - z_{\omega_{oq}}\right)}$$

(44)

$$\underbrace{\rho_n \left[\underbrace{e^{i\theta_{oq}} \left[\underbrace{ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}}}_{f_{oq}} - z_{\omega_{oq}} \right] + z_{\omega_{oq}}}_{\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})(z)} - z_{\omega_{oq}} \right] + z_{\omega_{oq}}}_{\mathcal{H}_{(\omega_{oq},\rho_n)}\left[\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})\right](z)} + C - C_0$$

$$t_{\overrightarrow{T_{oq}}}\left[\mathcal{H}_{(\omega_{oq},\rho_n)}\left[\mathcal{R}_{(\theta_{oq},\omega_{oq})}(f_{oq})\right]\right](z)$$

(45)

With,

$$\begin{cases} \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = \rho_n\left(z - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ \mathcal{R}_{(\theta_{oq},\omega_{oq})}(z) = e^{i\theta_{oq}}\left(z - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ f_{oq}(z) = ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}} \\ \overrightarrow{T_{oq}} = z + C + C_0 \end{cases}$$

(46)

Theorem 15. *If* $detM_{(1+2)} < 0$ *and* z_ω *and* $1-2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1-\bar{\mathcal{A}}) = 0$, *then the f complex application is the composite of* f_{oq} *oblique symmetry directed by the vector* $\overrightarrow{V_{oq}}\left(cos(\varphi_{oq}\alpha_{oq}),\ sin(\varphi_{oq}-\alpha_{oq})\right)$ *which makes the angle* α_{oq} *such as* $sin\ \alpha_{oq} = \frac{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}}{|\mathcal{A}|}$ *as for to an axis* Δ_{oq} *making the angle* φ_{oq} *as for to an axis* $(O,\ \vec{\imath})$ *by the rotation of the center* ω_{oq}, *the angle* θ_{oq} *such as* $\theta_{oq} = arg(\mathcal{A}) + \frac{\pi}{2}$ *and by homothetic of the* *center ω_{oq} and the ratio ρ_n and by translation with translation vector $\overrightarrow{T_{oq}}$ such as $z_{\overrightarrow{T_{oq}}}$ the affix of translation's.*

Example 3. *Let f be the complex application defined by :* $f(z) = iz + (1+i)\bar{z} + 2 + 3i$. *It is very easy to verify graphically that f is the composite of the oblique symmetry f_{oq} defined by: $f_{oq}(z) = -iz + (-1-i)\bar{z} - 4 - 4i$ directed by the vector $\vec{V}(-1,-1)$ which makes the angle $\alpha_{oq} = \frac{\pi}{4}$ relative to $\Delta_{oq} : x + 2 = 0$ axis, making the angle $\varphi_{oq} = \frac{\pi}{4}$ relative to $(O,\ \vec{\imath})$ axis, by the rotation of center $\omega_{oq}(-2,-1)$ and of angle $\theta_{oq} = \pi$ and the homothetic with ratio $\rho_n = 1$ and with center ω_{oq}, and by the translation $t_{\overrightarrow{T_{oq}}}$ an translation vector $\overrightarrow{T_{oq}}$ an affix $z_{\overrightarrow{T_{oq}}} = 2 + i$.*

It is very easy to verify directly that the image of point $M_0(0,\ 0)$ is point $M'_0(2,\ 3) = f(M_0)$. This result is well proven in Figure 7. It can also be verified using expressions (47).

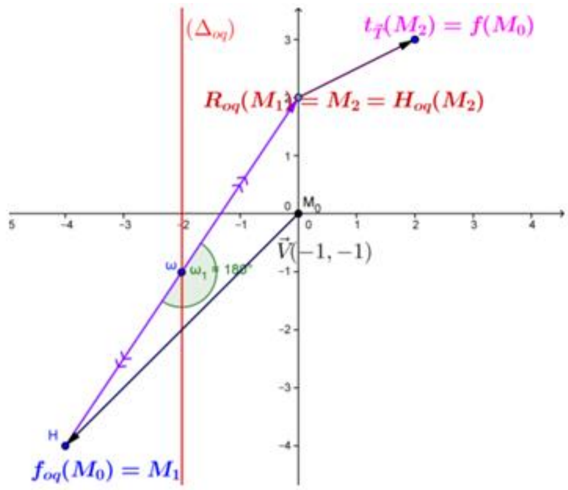

FIG. 7: Figure corresponding to Example 4.

$$\begin{cases} \overrightarrow{T_{oq}} = z + 2 + i \\ \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = z \\ \mathcal{R}_{(\theta_{oq},\omega_{oq})}(z) = -z - 4 - 2i \\ \quad f_{oq}(z) = -iz - (1+i)\bar{z} - 4 - 4i \quad \text{of axis } x + 2 = 0, \vec{V}(-1,-1) \end{cases}$$

(47)

From where, $f(z) = \overrightarrow{T_{oq}} o \mathcal{H}_{(\omega_{oq},\ \rho_n)}\, o\, \mathcal{R}_{(\omega_{oq},\theta_{oq})} o f_{oq}(z)$.

7.3. Case where $\boldsymbol{detM_{(1+2)} = 0}$

The $detM_{(1+2)} = 0$ means that f is not one-to-one. In other words, the system diffuses in the f complex application is bound. Hence the following theorem.

Theorem 19. *If* $detM_{(1+2)} = 0$, *then the f application is not one-to-one and its image is the line passing through* C *and* $\vec{U}_{A+B} = \overrightarrow{OA} + \overrightarrow{OB}$ *of director vector. That is* $Im(f) = \{M \in P/\overrightarrow{CM}//\vec{U}_{A+B} = \overrightarrow{OA} + \overrightarrow{OB}\}$.

Indeed,

$$\begin{cases} \underbrace{\mathcal{A}z + \mathcal{B}\bar{z} + C = z'}_{\Downarrow} \\ (a_1 + a_2)x + (b_2 - b_1)y = x' - x_0 \\ (b_1 + b_2)x + (a_1 - a_2)y = y' - y_0 \end{cases}$$

(48)

The main determinant denoted by Δ_x of this system is none other than $detM_{1+2} = |\mathcal{A}|^2 - |\mathcal{B}|^2$. This means that the two equations are homogeneous. Hence, this equation can still be written in the form below:

$$\begin{cases} [\mathcal{Re}(\mathcal{A}+\mathcal{B})]x + [Im(\mathcal{B}-\mathcal{A})]y = x' - \mathcal{Re}(C) \\ [Im(\mathcal{A}+\mathcal{B})]x + [\mathcal{Re}(\mathcal{A}-\mathcal{B})]y = y' - Im(C) \end{cases}$$

(49)

which is still equivalent to

$$[x' - \mathcal{Re}(C)][Im(\mathcal{A}+\mathcal{B})] - [y' - Im(C)][\mathcal{Re}(\mathcal{A}+\mathcal{B})] = 0.$$

From where:

$$\det\begin{vmatrix} x' - \mathcal{Re}(C) & \mathcal{Re}(\mathcal{A}+\mathcal{B}) \\ y' - Im(C) & Im(\mathcal{A}+\mathcal{B}) \end{vmatrix} = 0.$$

Example 5. *Let f be a complex application defined by :* $f(z) = (3+2i)z + (2-3i)\bar{z} + 2 + 2i$. *It is very easy checking that* $Im(f) = \{M \in P/\overrightarrow{CM}//\vec{U}_{A+B}: \ -x - 5y + 12 = 0\}$.

7.4. Case where $detM_{(1+2)} > 0$

This case confirms that $M_{(1+2)}$ is decomposing into a homothetic with ρ_p positive ratio and two isometrics one of which is positive and the other is negative. That is to say *f* can be written in the form: $f = f_{n_1} o f_{n_2}$, where $f_p = \mathcal{H}_{(\omega_{oq},\rho_p)} o \mathcal{R}_{(\theta_{oq},\omega_{oq})} o S_{\left(\Delta_{(oq,\ \omega_{oq},//x'ox)}\right)}$ and $f_{n_2} = f_{oq}$. Using the expression of $M_{(1+2)}$ and $M_{(q1+q2)}$, this composition is equivalent to $M_{(1+2)} = \rho_p \times M_{\theta_{oq}} \times M_{S_{\left(\Delta_{(oq,\ \omega_{oq},//x'ox)}\right)}} \times M_{(q1+q2)} = \rho_p \times M_{\theta_{oq}} \times K \times M_{(q1+q2)}$.

That this:

$$\begin{aligned} a_1 I + b_1 J + a_2 K + b_2 L &= \rho_p \left(I\cos\theta_{oq} + J\sin\theta_{oq} \right) K \left(Jb_{q1} + Ka_{q2} + Lb_{q2} \right) \\ &= \rho_p \left(I\cos\theta_{oq} + J\sin\theta_{oq} \right)\left(Ia_{q2} - Jb_{q2} - Lb_{q1} \right) \\ &= \rho_p [I\left(a_{q2}\cos\theta_{oq} + b_{q2}\sin\theta_{oq} \right) + J\left(a_{q2}\sin\theta_{oq} - b_{q2}\cos\theta_{oq} \right) + Kb_{q1}\sin\theta_{oq} - Lb_{q1}\cos\theta_{oq}] \end{aligned}$$

(50)

Indeed, this equation is of unknowns $\rho_n, b_{q1}, a_{q2}, b_{q2}, \theta_{oq}$. So that, by identification, we have:

$$\begin{cases} b_{q1}\sin\theta_{oq} = \dfrac{a_2}{\rho_p} \\ b_{q1}\cos\theta_{oq} = -\dfrac{b_2}{\rho_p} \\ a_{q2}\cos\theta_{oq} - b_{q2}\sin\theta_{oq} = \dfrac{a_1}{\rho_p} \\ a_{q2}\sin\theta_{oq} + b_{q2}\cos\theta_{oq} = \dfrac{b_1}{\rho_p} \end{cases}$$

(51)

The relation (49) leads us to have the relation (50) defined by:

$$\begin{cases} \rho_p = \sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2} \\ b_{q1} = -\dfrac{|\mathcal{A}|}{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}} \\ \sin\theta_{oq} = -\dfrac{\mathcal{Re}(\mathcal{B})}{|\mathcal{B}|} \\ \cos\theta_{oq} = \dfrac{Im(\mathcal{B})}{|\mathcal{B}|} \\ a_{q2} = -\dfrac{\mathcal{Re}(\mathcal{B})Im(\mathcal{A}) - Im(\mathcal{B})\mathcal{Re}(\mathcal{A})}{\rho_p|\mathcal{A}|} \\ a_{q2} = -\dfrac{\mathcal{Re}(\mathcal{A})\mathcal{Re}(\mathcal{B}) + Im(\mathcal{A})Im(\mathcal{B})}{\rho_p|\mathcal{A}|} \end{cases}$$

(52)

What's more,

$$\mathcal{A}\overline{\mathcal{B}} = \left[\mathcal{Re}(\mathcal{A}) + i\,Im(\mathcal{A})\right]\left[\mathcal{Re}(\mathcal{B}) - i\,Im(\mathcal{B})\right]$$
$$= \mathcal{Re}(\mathcal{A})\mathcal{Re}(\mathcal{B}) + Im(\mathcal{A})\,Im(\mathcal{B}) + i\left[\mathcal{Re}(\mathcal{B})\,Im(\mathcal{A}) - \mathcal{Re}(\mathcal{A})\,Im(\mathcal{B})\right]$$

(53)

On the other hand,

$$\cos\theta_{oq} + i\sin\theta_{oq} = \frac{Im(\mathcal{B})}{|\mathcal{B}|} - i\frac{\mathcal{Re}(\mathcal{B})}{|\mathcal{B}|} = -i\left(\frac{\mathcal{Re}(\mathcal{B})}{|\mathcal{B}|} + i\frac{Im(\mathcal{B})}{|\mathcal{B}|}\right)$$

(54)

The relation (49) leads us to have the relation (50) defined by:

$$\begin{cases} \rho_n = \sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2} \\ b_{q1} = -\dfrac{|\mathcal{B}|}{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}} \\ \sin\theta_{oq} = -\dfrac{\mathcal{Re}(\mathcal{B})}{|\mathcal{B}|} \\ \cos\theta_{oq} = \dfrac{Im(\mathcal{B})}{|\mathcal{B}|} \\ \theta_{oq} = \arg(\mathcal{B}) - \dfrac{\pi}{2} \\ a_{q2} = -\dfrac{Im(\mathcal{A}\overline{\mathcal{B}})}{\rho_p|\mathcal{B}|\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}} \\ a_{q2} = -\dfrac{\mathcal{Re}(\mathcal{A}\overline{\mathcal{B}})}{\rho_p|\mathcal{B}|\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}} \end{cases}$$

(55)

A- Case where *f* admits a fixed point

In the case where f admits a fixed point z_ω $(1 - 2\mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 \neq 0)$, and given the decomposition of $M_{(1+2)} = \rho_p \times M_{\theta_{oq}} \times M_{S_{\left(\Delta_{(oq,\ \omega_{oq},//x'ox)}\right)}} \times M_{(q1+q2)} = \rho_p \times M_{\theta_{oq}} \times K \times M_{(q1+q2)}$ leading to the result of the

expression (53), then we have: Let us recall in passing that:

$$\begin{cases} S_{\left(\Delta_{oq},\omega_{oq}//x'Ox\right)}(z)=\overline{\left(z-z_{\omega_{oq}}\right)}+z_{\omega_{oq}} \\ \mathcal{R}_{\left(\omega_{oq},\theta_{oq}\right)}(z)=e^{i\theta_{oq}}\left(z-z_{\omega_{oq}}\right)+z_{\omega_{oq}} \\ \mathcal{R}_{\left(\omega_{oq},\theta_{oq}\right)}oS_{\left(\Delta_{oq},\omega_{oq}//x'Ox\right)}(z)=S_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}x'Ox\right)}(z)=e^{i\theta_{oq}}\left(\overline{\left(z-z_{\omega_{oq}}\right)}+z_{\omega_{oq}}-z_{\omega_{oq}}\right)+z_{\omega_{oq}} \\ S_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}x'Ox\right)}of(z)=e^{i\theta_{oq}}\left(\overline{\left(f(z)-z_{\omega_{oq}}\right)}+z_{\omega_{oq}}-z_{\omega_{oq}}\right)+z_{\omega_{oq}} \\ \mathcal{H}_{\left(\omega_{oq},\rho_p\right)}(z)=\rho_p\left(z-z_{\omega_{oq}}\right)+z_{\omega_{oq}} \\ \mathcal{H}_{\left(\omega_{oq},\rho_p\right)}oS_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}x'Ox\right)}of(z)=\rho_p\left(e^{i\theta_{oq}}\left(\overline{\left(f(z)-z_{\omega_{oq}}\right)}+z_{\omega_{oq}}-z_{\omega_{oq}}\right)+z_{\omega_{oq}}-z_{\omega_{oq}}\right)+z_{\omega_{oq}} \\ \overrightarrow{T_{oq}}=z+C \\ \overrightarrow{T_{oq}}o\mathcal{H}_{\left(\omega_{oq},\rho_p\right)}oS_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}x'Ox\right)}of(z)=\rho_p\left(e^{i\theta_{oq}}\left(\overline{\left(f(z)-z_{\omega_{oq}}\right)}+z_{\omega_{oq}}-z_{\omega_{oq}}\right)+z_{\omega_{oq}}-z_{\omega_{oq}}\right)+z_{\omega_{oq}}+C \end{cases}$$

(56)

$$z'-z_{\omega}=\mathcal{A}\left(z-z_{\omega}\right)+\mathcal{B}\overline{\left(z-z_{\omega}\right)}$$

(57)

Given this relationship, for $\omega_{oq}=\omega$ we have:

$$\underbrace{\rho_n\left[\underbrace{e^{i\theta_{oq}}\left[\underbrace{ib_{q1}\left(z-z_{\omega_{oq}}\right)z+\left(a_{q2}+ib_{q2}\right)\overline{\left(z-z_{\omega_{oq}}\right)}+z_{\omega_{oq}}}_{f_{oq}}-z_{\omega_{oq}}\right]+z_{\omega_{oq}}-z_{\omega_{oq}}}_{S_{\left(\Delta_{oq},\omega_{oq}//x'Ox\right)}(f_{oq})(z)}\right]+z_{\omega_{oq}}}_{\mathcal{H}_{\left(\omega_{oq},\rho_n\right)}\left[S_{\left(\Delta_{oq},\omega_{oq}//x'Ox\right)}(f_{oq})\right](z)}$$

(58)

From where :

$$f(z)=\mathcal{H}_{\left(\omega_{oq},\rho_n\right)}oS_{\left(\Delta_{oq},\omega\frac{\theta_{oq}}{2}//x'Ox\right)}of_{oq}(z)$$

Theorem 14. *If* $detM_{(1+2)}>0$, *then for any given f complex application, the two angles* α_{oq} *and* φ_{oq} *associated with the* f_{oq} *complex application which is extracted from f are such as* $\alpha_{oq}=\frac{\sqrt{|\mathcal{B}|^2-|\mathcal{A}|^2}}{|\mathcal{B}|}$ *and* $\varphi_{oq}=\frac{\alpha}{2}+\frac{1}{2}arg(\bar{\mathcal{A}}\mathcal{B})-\frac{\pi}{2}$.

Theorem 15. *If* $detM_{(1+2)}<0$ *and* z_{ω} $(1-2\mathcal{Re}(\mathcal{A})+|\mathcal{A}|^2-|\mathcal{B}|^2\neq 0)$ *exists and is unique, then the f complex application is the composite of* f_{oq} *oblique symmetry directed by the vector* $\overrightarrow{V_{oq}}\left(cos\left(\varphi_{oq}-\alpha_{oq}\right),\ sin\left(\varphi_{oq}-\alpha_{oq}\right)\right)$ *which makes the angle* α_{oq} *such as* sin $\alpha_{oq}=$

$\frac{\sqrt{|\mathcal{A}|^2-|\mathcal{B}|^2}}{|\mathcal{B}|}$ *as for to an axis* Δ_{oq} *making the angle* φ_{oq} *as for to an axis* $(O,\ \vec{\imath})$ by the orthogonal symmetry $S_{\left(\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox\right)}\right)}$ compared to the axis $\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox\right)}$ with equation $\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox\right)}:\ \left(x-x_{\omega_{oq}}\right)sin\frac{\theta_{oq}}{2}-\left(x-y_{\omega_{oq}}\right)cos\frac{\theta_{oq}}{2}$ and by homothetic of the center ω_{oq} and the ratio ρ_p.

Example 2. *Let f be the complex application defined by :* $f(z)=(-2+2i)z-2i\bar{z}+3$. *It is very easy to verify graphically that f is the composite of the oblique symmetry foq defined by:* $f_{oq}(z)=-iz+(1+i)\bar{z}$ *directed by the vector* $\vec{V}(1,-1)$ *which makes the angle* $\alpha_{oq}=\frac{\pi}{4}$ *relative to* $\Delta_{oq}: y=0$ *axis, making the angle* $\varphi_{oq}=\frac{\pi}{4}$ *relative to* $(O,\ \vec{\imath})$ *axis, by the orthogonal symmetry as for to an axis* $\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox\right)}: x-1=0$ *and the homothetic with ratio* $\rho_n=2$ *and with center* $\omega_{oq}(1,0)$.

It is very easy to verify directly that the image of point $M_1(1,\ 1)$ is point $M'_1(-3,\ 2)=f(M_1)$. This result is well proven in Figure 8. It can also be verified using expressions (59).

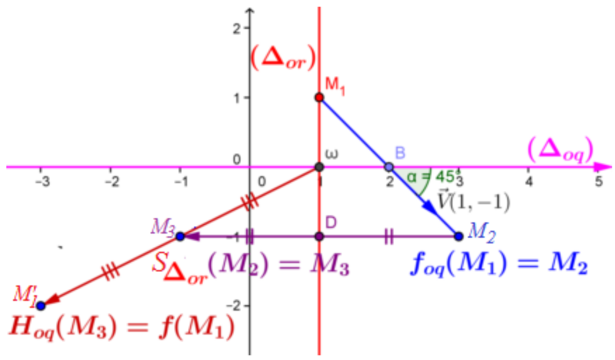

FIG. 8: Figure corresponding to Example 6

$$\begin{cases} \mathcal{H}_{\left(\omega_{oq},\rho_n\right)}(z)=2z-1 \\ S_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}//x'Ox\right)}(z)=-\bar{z}+2 \quad \text{of axix } x-1=0 \\ f_{oq}(z)=-iz+(1+i)\bar{z} \quad \text{of axis } y=0\,;\vec{V}(-1,1) \end{cases}$$

(59)

From where, $f(z) = \mathcal{H}_{(\omega_{oq},\ \rho_n)}\, o\, S_{\left(\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)}\right)} o f_{oq}(z).$

B- Case where $\mathbf{1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0}$ *and* $\mathbf{\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1-\bar{\mathcal{A}}) = 0}$

As we have already announced in paragraph 4, these two cases tell us that the f complex application admits an infinity of fixed points. It is then clear from **Corollary 2** that f has the set of points invariants of which $Inv(f_{oq})$ defined by $nv(f): x[1 - \mathcal{R}e(\mathcal{A}+\mathcal{B})] + y[Im(\mathcal{A}-\mathcal{B})] - x[Im(\mathcal{A}+\mathcal{B})] + y[1 - \mathcal{R}e(\mathcal{A}-\mathcal{B})] = \mathcal{R}e(\mathcal{C}) + Im(\mathcal{C})$. We can then choose one of this bridge whose ω_{oq}. Which means that:

$$z' - z_{\omega_{oq}} = \mathcal{A}\left(z - z_{\omega_{oq}}\right) + \mathcal{B}\overline{\left(z - z_{\omega_{oq}}\right)}$$

(60)

$$\underbrace{\rho_n\left[\underbrace{e^{i\theta_{oq}}\left[\overline{\underbrace{ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}}}_{f_{oq}}} - z_{\omega_{oq}}\right] + z_{\omega_{oq}}}_{S_{\left(\Delta_{oq},\omega_{oq}//x'Ox\right)}(f_{oq})(z)} - z_{\omega_{oq}}\right] + z_{\omega_{oq}}}_{\mathcal{H}_{\left(\omega_{oq},\rho_n\right)}\left[S_{\left(\Delta_{oq},\omega_{oq}//x'Ox\right)}(f_{oq})\right](z)}$$

(61)

From where :

$$f(z) = \mathcal{H}_{\left(\omega_{oq},\rho_n\right)} o S_{\left(\Delta_{oq},\omega\frac{\theta_{oq}}{2}//x'Ox\right)} o f_{oq}(z)$$

Theorem 15. *If* $detM_{(1+2)} < 0$ *and* z_ω *and* $1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1-\bar{\mathcal{A}}) = 0$, *then the f complex application is the composite of* f_{oq} *oblique symmetry directed by the vector* $\overrightarrow{V_{oq}}\left(cos(\varphi_{oq} - \alpha_{oq}),\ sin(\varphi_{oq} - \alpha_{oq})\right)$ *which makes the angle* α_{oq} *such as* $\sin \alpha_{oq} = \frac{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}}{|\mathcal{B}|}$ *as for to an axis* Δ_{oq} *making the angle* φ_{oq} *as for to an axis* $(O,\ \vec{\imath})$ *by the orthogonal symmetry* $S_{(\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)})}$ *compared to the axis* $\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)}$ *with equation* $\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)}$: $\left(x - x_{\omega_{oq}}\right) sin\frac{\theta_{oq}}{2} - \left(x - y_{\omega_{oq}}\right) cos\frac{\theta_{oq}}{2}$ *and by homothetic of the center* ω_{oq} *and the ratio* ρ_p.

Example 2. *Let f be the complex application defined by:* $f(z) = (1+i)z + i\bar{z}$. *It is very easy to verify graphically that f is the composite of the oblique symmetry foq defined by:* $f_{oq}(z) = -iz + (1-i)\bar{z}$ *directed by the vector* $\vec{V}(0,1)$ *which makes the angle* $\alpha_{oq} = \frac{\pi}{4}$ *relative to* Δ_{oq} : $y = -x$ *axis, making the angle* $\varphi_{oq} = \frac{\pi}{4}$ *relative to* $(O,\ \vec{\imath})$ *axis, by the orthogonal symmetry as for to an axis* $\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x\prime ox\right)}$: $y = 0$ *and the homothetic with ratio* $\rho_n = 1$ *and with center* $\omega_{oq}(0,0)$.

It is very easy to verify directly that the image of point $M_1(1,\ 1)$ is point $M'_1(-3,\ 2) = f(M_1)$. This result is well proven in Figure 9. It can also be verified using expressions (63).

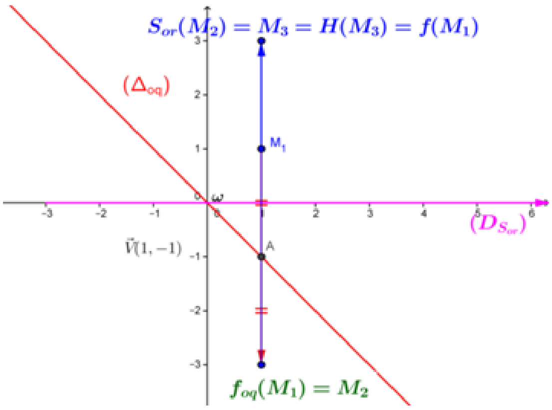

FIG. 9: Figure corresponding to Example 7

$$\begin{cases} \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = z \\ S_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}//x'Ox\right)}(z) = \bar{z} \quad \text{of axix } y = 0 \\ f_{oq}(z) = -iz + (1+i)\bar{z} \quad \text{of axis } y = -x\,;\vec{V}(0,1) \end{cases}$$

(63)

From where, $f(z) = \mathcal{H}_{(\omega_{oq},\ \rho_n)}\ o\ S_{\left(\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x\prime ox\right)}\right)} o f_{oq}(z)$.

C- Case where $\mathbf{1 - 2\mathcal{Re}(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0}$ *and* $\mathbf{\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1-\bar{\mathcal{A}}) \neq 0}$

Paragraph 4 has already confirmed that this case corresponds to the non-existence of a fixed point of the corresponding f complex application. In this case, we are going to construct a complex constant $\mathcal{C}_o$ so that $\mathcal{B}\bar{\mathcal{C}}_o - \mathcal{C}_o(1-\bar{\mathcal{A}}) = 0$. It is then clear from **Corollary 2** that f has the set of invariant points whose $nv(f)$ defined by $nv(f)\colon x[1 - \mathcal{Re}(\mathcal{A}+\mathcal{B})] + y[Im(\mathcal{A}-\mathcal{B})] - x[Im(\mathcal{A}+\mathcal{B})] + y[1 - \mathcal{Re}(\mathcal{A} -$

$\mathcal{B})] = \mathcal{R}e(\mathcal{C}_o) + Im(\mathcal{C}_o)$. We can then choose one of this bridge whose ω_{oq}. Hence the theorem following.

Theorem 17. *For any f complex application, if* $1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) \neq 0$, *then there exists* $\mathcal{C}_o = ik(1 + \mathcal{B} - \mathcal{A})$ *where* $k \in \mathbb{R}^*$ *verifying* $\mathcal{B}\bar{\mathcal{C}}_o - \mathcal{C}_o(1 - \bar{\mathcal{A}}) = 0$ *such as* $f(z) = \mathcal{A}z + \mathcal{B}\bar{z} + \mathcal{C}_o + \mathcal{C} - \mathcal{C}_o$.

Given the expression of $f(z) = \mathcal{A}z + \mathcal{B}\bar{z} + \mathcal{C}_o + \mathcal{C} - \mathcal{C}_o$ we have:

$$z' - z_{\omega_{oq}} = \mathcal{A}\left(z - z_{\omega_{oq}}\right) + \mathcal{B}\overline{\left(z - z_{\omega_{oq}}\right)}$$

(64)

$$\underbrace{\underbrace{\rho_n \left[\underbrace{e^{i\theta_{oq}} \left[\overline{\underbrace{ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}}}_{f_{oq}}} - z_{\omega_{oq}} \right] + z_{\omega_{oq}} - z_{\omega_{oq}}}_{S_{\left(\Delta_{oq}, \omega_{oq}, \frac{\theta_{oq}}{2} // x'Ox\right)}(f_{oq})(z)} \right] + z_{\omega_{oq}}}_{\mathcal{H}_{(\omega_{oq}, \rho_n)}\left[S_{\left(\Delta_{oq}, \omega_{oq}, \frac{\theta_{oq}}{2} // x'Ox\right)}(f_{oq})\right](z)} + C - C_0}_{t_{\overrightarrow{T_{oq}}}\left[\mathcal{H}_{(\omega_{oq}, \rho_n)}\left[S_{\left(\Delta_{oq}, \omega_{oq}, \frac{\theta_{oq}}{2} // x'Ox\right)}(f_{oq})\right]\right](z)}$$

(65)

With,

$$\begin{cases} \mathcal{H}_{(\omega_{oq}, \rho_n)}(z) = \rho_n\left(z - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ S_{\left(\Delta_{oq}, \omega_{oq}, \frac{\theta_{oq}}{2} x'Ox\right)}(z) = e^{i\theta_{oq}}\left(\overline{\left(z - z_{\omega_{oq}}\right) + z_{\omega_{oq}}} - z_{\omega_{oq}}\right) + z_{\omega_{oq}} \\ f_{oq}(z) = ib_{q1}\left(z - z_{\omega_{oq}}\right)z + \left(a_{q2} + ib_{q2}\right)\overline{\left(z - z_{\omega_{oq}}\right)} + z_{\omega_{oq}} \\ \overrightarrow{T_{oq}} = z + C + C_0 \end{cases}$$

(66)

Theorem 15. *If* $detM_{(1+2)} < 0$ *and* z_ω *and* $1 - 2\mathcal{R}e(\mathcal{A}) + |\mathcal{A}|^2 - |\mathcal{B}|^2 = 0$ *and* $\mathcal{B}\bar{\mathcal{C}} - \mathcal{C}(1 - \bar{\mathcal{A}}) = 0$,, *then the f complex application is the composite of* f_{oq} *oblique symmetry directed by the vector* $\overrightarrow{V_{oq}}\left(cos(\varphi_{oq} - \alpha_{oq}),\ sin(\varphi_{oq} - \alpha_{oq})\right)$ *which makes the angle* α_{oq} *such as* sin $\alpha_{oq} = \frac{\sqrt{|\mathcal{A}|^2 - |\mathcal{B}|^2}}{|\mathcal{B}|}$ *as for to an axis* Δ_{oq} *making the angle* φ_{oq} *as for to an axis* $(O,\ \vec{\imath})$ *by the orthogonal symmetry* $S_{(\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)})}$ *compared to the axis* $\Delta_{(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x'ox)}$ *with* *equation* $\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x\prime ox\right)}: \left(x - x_{\omega_{oq}}\right) sin\frac{\theta_{oq}}{2} - \left(x - y_{\omega_{oq}}\right) cos\frac{\theta_{oq}}{2}$ *and by homothetic of the center* ω_{oq} *and the ratio* ρ_p.

Example 2. *Let f be the complex application defined by :* $f(z) = (1+i)z + i\bar{z}$ *+4-3i. It is very easy to verify graphically that f is the composite of the oblique symmetry foq defined by:* $f_{oq}(z) = -iz + (1-i)\bar{z}$ *directed by the vector* $\vec{V}(0,1)$ *which makes the angle* $\alpha_{oq} = \frac{\pi}{4}$ *relative to* Δ_{oq} : $y = -x$ *axis, making the angle* $\varphi_{oq} = \frac{\pi}{4}$ *relative to* $(O,\ \vec{\imath})$ *axis, by the orthogonal symmetry as for to an axis* $\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x\prime ox\right)}: y = 0$ *and the homothetic with ratio* $\rho_n = 1$ *and with center* $\omega_{oq}(0{,}0)$, *and by the translation* $t_{\overrightarrow{T_{oq}}}$ *an translation vector* $\overrightarrow{T_{oq}}$ *an affix* $z_{\overrightarrow{T_{oq}}} = 4 + 3i$.

.It is very easy to verify directly that the image of point $M_1(1,\ 1)$ is point $M'_1(-3,\ 2) = f(M_1)$. This result is well proven in Figure 10. It can also be verified using expressions (67).

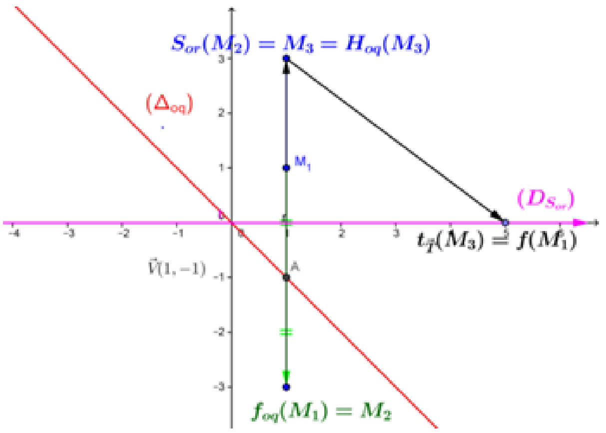

FIG. 7: Figure corresponding to Example 4.

$$\begin{cases} \overrightarrow{T_{oq}} = z + 4 + 3i \\ \mathcal{H}_{(\omega_{oq},\rho_n)}(z) = z \\ S_{\left(\Delta_{oq},\omega_{oq},\frac{\theta_{oq}}{2}//x'Ox\right)} = \bar{z} \quad \text{of axis } \ y = 0 \\ \quad f_{oq}(z) = -iz + (1-i)\bar{z} \quad \text{of axis } y = -x, \vec{V}(0,1) \end{cases}$$

(67)

From where, $f(z) = \mathcal{H}_{(\omega_{oq},\ \rho_n)}\ o\ S_{\left(\Delta_{\left(oq,\ \omega_{oq},\ \frac{\theta_{oq}}{2}\%x\prime ox\right)}\right)} of_{oq}(z$

8. Application

In a plan provided with an orthonormal reference $(O,\ \vec{\imath},\ \vec{\jmath})$, we consider an oblique symmetry of axis $\Delta\,;x+y-3=0$ of direction vector $\vec{U}(-1\ ,\ 1)$ making an angle $\varphi=\frac{3\pi}{4}$ compared to the axis $(O,\vec{\imath})$, directed by a vector $\overrightarrow{V'}(a'\ ,\ b')$ making an angle $\alpha=\frac{\pi}{3}$ compared to Δ.

- **Exercise 1**

- Give the coordinates $\overrightarrow{V'}(a'\ ,\ b')$ of vectors depending on φ and α.
- Give the analytical and complex expression of this symmetry. Check if this expression represents oblique symmetry or not.
- Give these expression if $\alpha=\frac{\pi}{2}$

- **Exercise 2**

Let f_1, f_2 , f_3 , f_4 , f_5 and f_6 be the complex maps defined by:

$$\begin{cases} f_1(z)=\left(\sqrt{3}-i\right)z+\left[\sqrt{3}+1-i\left(\sqrt{3}-1\right)\right]\bar{z} \\ f_2(z)=iz+\left(1+i\right)\bar{z}+4i \\ f_3(z)=iz+\left(1+i\right)\bar{z}+1+4i \\ f_4(z)=\left(-3+3i\right)z-3i\bar{z}+6 \\ f_5(z)=\left(1+i\right)z+i\bar{z}-2+4i \\ f_6(z)=-iz-\left(1-i\right)\bar{z} \end{cases}$$

Characterize the six complex applications indicated above.

9. Conclusion and Perspectives

In this paper, we found that the meaning of geometric propositions is to translate situations geometric by scenarios between properties in mathematics. This is the case of the theorem of several elaborations of the models of the entire theorem existing in the literature, which translates the situations of the sum, geometric elements. It is for the benefit of the meaning of what is taught that oblique symmetry and its fields of application must have their place in education. These thoughts on teaching are based on an analysis of curricula, textbooks and teaching practices. But they are also based on a historical reading of mathematical texts. It is also important to notice that this work has allowed us to solve problems of general, analytical and complex expression, of a symmetry oblique generating the general, analytical and complex expression of an orthogonal symmetry. We will think to try to emit some reflections on this approach on the symmetries of molecules and crystals based especially on a molecule or a crystal which is defined in quantum mechanics by the wave function of all its constituents, nuclei and electrons. But for many uses, one can model the structure simply by the positions of the centers of atoms or ions. The notions of symmetry of the molecule or the crystal can indeed be defined as the isometrics of the space which are also automorphisms for the monadic predicates of structure. Also, we will consider in its application on the extension of the notions of the studies on periodic trajectories (oscillations, vibrations, satellite movements) which are structures spatio-temporal symmetric for certain translations in time.

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Author contributions

The author(s) declare that all author(s) contributed equally,

approve this work, and take full responsibility.

Data Availability

No data were used to support the results of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Institutional review board statement

Not applicable.

Informed consent statement

The author(s) declare that human participants were involved in this study, and have provided all relevant informed consent forms.

Sample availability

The author(s) declare that no physical samples were used in this study.

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